HOMOGENEOUS STRONGLY PSEUDOCONVEX HYPERSURFACES

*by Hugo Rossi**

§1. *Introduction*

Let $L \xrightarrow{\tau} E$ be a negative complex line bundle over the algebraic variety E. Suppose that G is a compact group of holomorphic bundle transformations of L. If the induced action of G on E is transitive, then the typical orbit of G on L is a hypersurface (since L is negative, it is not topologically trivial). These orbits are examples of homogeneous strongly pseudoconvex hypersurfaces — a notion to be defined below. In this paper we shall discuss the extent to which these are the only examples. The following collection of definitions of CR-structure was first made by Greenfield [5].

1.1 **Definition.** Let M be a C^∞ manifold, and B a subbundle of the tangent bundle $T(M)$. Let $J: B \to B$ be a bundle automorphism such that $J^2 = -I$. The triple (M, B, J) is a *CR-manifold* if the sheaf of complex vector fields

$$\mathbf{A}(M) = \{X - iJX;\ X \in \mathbf{B}\}$$

is closed under Lie brackets.

Remark. We may complexify J to $J_{\mathbf{C}}: B_{\mathbf{C}} \to B_{\mathbf{C}}$. Here $B_{\mathbf{C}}$ splits into the sum of two bundles, $B_{\mathbf{C}} = A + \bar{A}$, where $J_{\mathbf{C}} | A =$ multiplication by i. $\mathbf{A}$ is just the sheaf of sections of A.

1.2 **Definition.** If B is of codimension one in $T(M)$, we say that (M, B, J) is a *CR-hypersurface*.

In this case we can find, locally, a form $\eta \in T^*(M)$ such that $B = \eta^{\perp}$. It is easily verified (cf. [5]) that the hermitian form on A,

$$L(X, Y) = \mathrm{id}\,\eta(X, \bar{Y})$$

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is well-defined up to a multiplicative constant. (Notice, if $\hat{X}, \hat{Y}$ are vectoı fields in A through X, Y respectively,

$$d\eta(X, \bar{Y}) = -\eta([X, \hat{Y}]).$$

1.3 **Definition.** Let (Σ, B, J) be a CR-hypersurface. We say that Σ is *strongly pseudoconvex* if the *Levi form* L is definite (all eigenvalues of the same sign).

1.4 **Definition.** Let (M, B, J), (M', B', J') be CR-manifolds, and $\phi: M \to M'$ a C^∞ mapping. ϕ is a *CR-map* if $d\phi: B_p \to B'_{\phi(p)}$ for all $p \in M$, and $d\phi \circ J = J' \circ d\phi$. The *sheaf of CR-functions* $\mathcal{O}(M)$ is the sheaf of CR-maps $\phi: (M, B, J) \to (\mathbf{C}, T(\mathbf{C}), i)$.

A real hypersurface in a complex manifold inherits a CR-structure (see [5] again) called the induced CR-structure. Only in the real analytic category is it known that a CR-structure is locally induced;[1] for this reason we shall, in this paper, restrict attention to the category of real-analytic CR-manifolds and maps. In this case, the CR-functions are (locally) the restrictions of holomorphic functions in the ambient complex manifold (and for this reason shall sometimes be called holomorphic).

1.5 **Definition.** Let (M, B, J) be a CR-manifold. $G(M)$ is the group of CR-bimorphisms of M. If G is a Lie group and $G \to G(M)$ is a representation of G in $G(M)$ so that the action of G is transitive, we shall say that M is a G-CR-manifold.

We shall need the following fact (for a proof for hypersurfaces, see [11]).

1.6 **Proposition.** *Let (M, B, J) be a CR-manifold, and ξ a real vector field on M. ξ generates locally a one-parameter group of CR-bimorphisms if and only if for every $v \in \mathbf{A}$, $[\xi, v] \in \mathbf{A}$ also.*

1.7 **Theorem.** [3, 11, 12] *If (M, B, J) is a CR-hypersurface with non-degenerate Levi form, then $G(M)$ is a Lie group.*

The case $\dim M = 3$ is due to E. Cartan; higher dimensions to N. Tanaka. The following, finally, is a basic concept:

1.8 **Definition.** An $H\Sigma$-manifold is a compact, G-CR-strongly pseudoconvex manifold for some group G.

Because of Theorem 1.7, we may assume G is a Lie group. We now set out to identify all $H\Sigma$-manifolds. First, we have the theorem of Morimoto and Nagano:

1.9 **Theorem** [8]. *Let Σ be a simply connected $H\Sigma$-manifold contained in a Stein manifold. Then Σ is either the sphere in $\mathbf{C}^n$, or the unit tangent sphere bundle to a compact simply connected symmetric space B of rank 1.*

The situation we consider is that in which Σ is *not* contained in any Stein manifold, simply connected or not. We stress that we impose this condition so that the case of an $H\Sigma$-submanifold of a Stein manifold, with a non-compact universal cover, remains open (outside of some further exceptions in dimension 3). These are the results:

Theorem (3.3). *An $H\Sigma$- space of dimension $\geqq 5$ is the boundary of a normal Stein analytic space with at most one singular point.*

Theorem (4.1). *Let D be a strongly pseudoconvex neighborhood of an isolated singular point $\{p_0\}$. Then $G(D)$ is compact. Suppose $G(D)$ has a hypersurface orbit Σ. Then Σ is an $H\Sigma$-space. Let $\hat{D} \xrightarrow{\pi} D$ be the monoidal transformation of $\{p_0\}$, $E = \pi^{-1}(p_0)$. Then $\hat{D}, E$ are non-singular, $G(D)$ lifts to $\hat{D}$, is transitive on E, and the other orbits of $G(D)$ are hypersurfaces.*

Theorem (6.1). *Let Σ be an $H\Sigma$-manifold bounding a Stein space with nonempty singular set. Then Σ is a circle bundle in a homogeneous line bundle over a homogeneous algebraic variety E.*

Now, to what extent do these theorems describe all $H\Sigma$-manifolds?

A. dimension $\Sigma \geqq 5$. Then Σ bounds a Stein space X.

(1) if X has singularities, Theorem (6.2) is the description.

I further suspect that E is a C-space, and $G(X)$ is a compact semi-simple Lie group.[2]

(2) if X is a manifold, and $\pi_1(\Sigma) < \infty$, we can lift the CR-structure of Σ to its universal cover $\tilde{\Sigma}$, which will again be an $H\Sigma$-manifold. By Theorem (2.3), $\tilde{\Sigma}$ bounds some $\tilde{X}$, and the projection map $\pi : \tilde{\Sigma} \to \Sigma$ extends to a holomorphic map $\pi : \tilde{X} \to X$. The branching locus of π is a compact analytic subvariety of X, so is a discrete set. Since X has no singularities, and $\dim X \geqq 3$, π can have no branch points, so $\tilde{X}$ is also a Stein manifold. By the theorem of Morimoto and Nagano, $\tilde{X}$ is known, and X is just the quotient of $\tilde{X}$ by a finite group.

(3) the remaining case is that of Σ in a Stein manifold with $\pi_1(\Sigma)$ infinite. I know no examples.

B. dimension $\Sigma = 3$.

(1) if Σ bounds a Stein space X so that $G(\Sigma)$ has a fixed point on X, then we can conclude that Σ is the unit sphere in $\mathbf{C}^2$ modulo a finite subgroup of $SU(2)$ (see Theorem 5.1).

(2) if Σ bounds a Stein space *with* singularities, we are in the first case again.

(3) if Σ bounds a Stein manifold, and is simply connected, the theorem of Morimoto and Nagano applies.

(4) Σ need not bound any analytic space.

(5) if Σ bounds a Stein manifold, and $\pi_1(\Sigma) < \infty$, the conclusion of Theorem 6.2 may still fail.

The examples for (4) and (5) are given by the images of the unit sphere in $\mathbf{C}^2$ in the spaces M, $\tilde{Q}$ of Theorem 6.1 [*10*]. $SU(2)$ acts holomorphically on these spaces since it leaves the form $dz_1 \wedge dz_2 + \varepsilon \partial \bar{\partial} \log(z_1 \bar{z}_1 + z_2 \bar{z}_2)$ (which defines the complex structure) invariant.

§2. *Some preliminary results*

In this section we collect some necessary facts which are easily obtainable from the definitions, or the literature.

2.1. **Theorem.** *Let (M, B, J) be a compact real analytic CR-manifold. There is a complex manifold X and a CR-injection $j: M \to X$ such that M carries the induced structure.*

Proof. Let $d = \dim M$, $2n = \dim B$. Let $\tilde{M}$ be a complexification of M. For $p \in M$, we can find real analytic vector fields $X_1, \cdots, X_n$ spanning A in a neighborhood U_p of p. We may consider these as the restrictions to U_p of holomorphic vector fields $\tilde{X}_1, \cdots, \tilde{X}_n$ defined in a neighborhood $\tilde{U}_p$ of p in $\tilde{M}$. Since the linear span of the $\tilde{X}_j$ is closed under brackets, by the Frobenius theorem, we can find holomorphic functions $z_1, \cdots, z_{d-n}$ defined in a neighborhood of p (which we take to be $\tilde{U}_p$) which determine the vector fields $\tilde{X}_i$ is this sense: if f is holomorphic in $W \subset \tilde{U}_p$, $\tilde{X}_j f = 0$, $1 \leqq j \leqq n$ if and only if f is a holomorphic function of $z_1, \cdots, z_{d-n}$. In particular, if f is real-analytic on M and $X_i f = 0$, f is the trace of a holomorphic function of $z_1, \cdots, z_{d-n}$.

We can cover M by finitely many such neighborhoods $\tilde{U}_\alpha$, with corresponding functions $z_1^\alpha, \cdots, z_{d-n}^\alpha$. In $\tilde{U}_\alpha \cap \tilde{U}_\beta$, z^α is a holomorphic function of $z^\beta : z^\alpha = F_\beta^\alpha(z^\beta)$. We now take $M \subset \cup\, \tilde{U}_\alpha$ small enough so that each point $p \in \tilde{M}$ lies on a sheet of the fibration defined by the $\tilde{X}_j$. These sheets intersect M in a manifold whose tangent space is the set of real vectors in the span of $X_1, \cdots, X_n$; but this is zero-dimensional. Thus, we can take $\tilde{M}$ so small that each sheet intersects M in a single point. Let $X = \tilde{M}/E$, where E is the equivalence relation: pEq if and only if p, q lie on the same sheet. Clearly X is a complex manifold with coordinate neighborhoods

$(\tilde{U}_\alpha/E, z_1^\alpha, \cdots, z_{d-n}^\alpha)$ and transition functions F_β^α. If $j: \tilde{M} \to X$ is the quotient map, $j: M \to X$ is the required injection.

In the case we are considering, M is a strongly pseudoconvex hypersurface. Coupling the above with the results of [*10*], we obtain

2.2 **Theorem.** *Let Σ be a strongly pseudoconvex hypersurface, $\dim\Sigma \geqq 5$. Then there is a unique strongly pseudoconvex normal Stein analytic space $X \subset\subset \hat{X}$ with only isolated singular points, and a CR-injection $j: \Sigma \to \hat{X}$ of Σ onto ∂X.*

Proof. Take $\hat{X}$ the X of Theorem 2.1 together with its hole filled in. In [*10*], an example was given of a Σ of dimension 3 which did not bound any analytic space. As pointed out in the introduction, that example is $H\Sigma$; here we give another class of examples. Suppose that $(SU(2), B, J)$ is an $H\Sigma$-manifold under left translation by the group $SU(2)$. If $SU(2)$ bounds a Stein manifold, since it is simply connected, by the theorem of Morimoto and Nagano, $(SU(2), B, J)$ can be realized (as a CR-manifold) as the sphere in $\mathbf{C}^2$, under the usual action of the unitary group. If $SU(2)$ bounds a Stein space X with singularities, we shall see that the action of $SU(2)$ extends to X with a unique fixed point $\{p_0\}$ (Proposition 3.3). By Theorem 5.1 we will conclude that p_0 is in fact a regular point, and $SU(2)$ is once again the unit sphere. Thus this is the only possibility for $SU(2)$ to be an $H\Sigma$-manifold bounding an analytic space (which will be necessarily Stein by pseudo-convexity).

Now let V_1, V_2, V_3 be a cyclic basis for $SU(2)$: $[V_i, V_j] = V_k$ cyclically. Letting B be the distribution spanned by V_1, V_2 (considered as left-invariant vector fields on $SU(2)$) so that the "holomorphic tangent space" is spanned by $A = V_1 + iV_2$, the resulting $H\Sigma$-space is that of the unit sphere in $\mathbf{C}^2$. For, since $[V_3, A] = 2iA$, and Γ is the group generated by V_3, $SU(2)/\Gamma$ inherits a complex structure, and is obviously $\mathbf{P}^1$. The projection $\pi: SU(2) \to \mathbf{P}^1$ is a CR-map.

However, if we choose $A = V_1 + i(V_2 + V_3)$, there is no nonzero vector $Z \in SU(2)$ such that $[Z, A] \in \mathbf{C}A$, so the resulting $H\Sigma$-manifold could not be the sphere in $\mathbf{C}^2$, and thus cannot be a boundary. Suppose for $Z = \sum a_i V_i$, $a_i \in \mathbf{R}$, $[Z, A] = cA, c \in \mathbf{C}$. This gives

$$i(a_2 - a_3)V_1 + (a_3 - ia_1)V_2 + (ia_1 - a_2)V_3 = cV_1 + ic(V_2 + V_3),$$

so $c = i(a_2 - a_3)$ is pure imaginary, and thus $a_1 = 0$, $a_3 = -a_2 = ic$. Thus $a_2 = a_3 = c = 0$ also.

Finally, we will need the following criterion for a CR-torus bundle $\tau: F \to M$ to be included in a holomorphic vector bundle.

2.3. **Proposition.** *Let $\tau: F \to M$ be a torus bundle, where M is a complex manifold and (F, B, J) is a CR-manifold such that*

(a) *the torus action is CR,*

(b) *τ is CR,*

(c) $d\tau\colon B_p \cong B_{\tau(p)} = T_{\mathbf{C}}(M)_{\tau(p)}$.

Then there exist a principal $(\mathbf{C}^*)^n$-bundle $\pi: P \to M$, and a CR-map $j: F \subset P$ such that $\tau = \pi \circ j$. If F, M are G-manifolds, τ a G-map, then P is a G-manifold and π, j are G-maps.

Proof. Let $\theta_1, \cdots, \theta_k$ be the vector fields on F generating the torus action on $\tau: F \to M$. Let $Z_0 \subset T_{\mathbf{C}}(F)$ be the bundle of holomorphic vectors. By (a), the operations $[\theta_j, \cdot\,]$ induce endomorphisms of the sheaf A_0. Let $P = F \times (\mathbf{R}^+)^n$, and $\pi: P \to M$, $\pi(f, r) = \tau(f)$, $j: F \to P$, $j(f) = (f, 1)$. Since $T_{\mathbf{C}}(P) = T_{\mathbf{C}}(F) + \mathbf{R}^n$ in a canonical way, we can consider $A_0 \subset T_{\mathbf{C}}(F)$ as a subbundle of $T_{\mathbf{C}}(P)$. Let A be the subbundle of $T_{\mathbf{C}}(P)$ generated by A_0 and

$$X_j = \frac{i}{r_j}\theta_j + \frac{\partial}{\partial r_j}, \quad 1 \leqq j \leqq k.$$

Then, by (c), $T(F) = \bigoplus_j \mathbf{R}\theta_j \oplus B$, so

$$\begin{aligned} T_{\mathbf{C}}(P) &= \bigoplus_j \mathbf{C}\frac{\partial}{\partial r_j} \oplus \mathbf{C}\theta_j \oplus A_0 \oplus \bar{A}_0 \\ &= \bigoplus_j \mathbf{C}\left(\frac{\partial}{\partial r_j} + \frac{i}{r_j}\theta_j\right) \oplus A_0 \oplus \overline{\cdots} \\ &= A \oplus \bar{A}, \end{aligned}$$

so A is the bundle of $(1,0)$ vectors in some almost complex structure on P. But this structure is integrable, for A is a Lie algebra sheaf under brackets: $[X_j, V_0] = i/r_j[\theta_j, V_0]$ is in A_0 for $V_0 \in A_0$, and if $V_1 \in A_0$, so is $[V_0, V_1]$. Thus P is a complex manifold, and because of (b) π is holomorphic. The fibers of π are biholomorphic to $(\mathbf{C}^*)^n$, and the vector fields X_j are global holomorphic vector fields tangent to the fibers, so generate the $(\mathbf{C}^*)^n$ action exhibiting $\pi: P \to M$ as a $(\mathbf{C}^*)^n$ bundle. The map j is clearly a CR-injection $(j^*(A) = A_0)$. It is obvious that a given CR-bundle action of a group G on F extends holomorphically to P.

§3. *Compactness of the group*

The following result is probably true in greater generality (using the Bergman metric); in our case the proof is particularly simple.

3.1 **Lemma.** *Let X be a strongly pseudoconvex Stein space, and G the group of automorphisms of X. If G has a relatively compact orbit, G is compact.*

Proof. Let O be the orbit, and $p_0 \in O$. For $x \in X$, define

$$m(x) = \sup\{|f(x)|;\, f \in \mathcal{O}(X),\ p \in O,\, f(p) = 0,\ \|f\|_\infty \leqq 1\}$$

Let $E_m = \{x \in X;\ m(x) \leq 1 - m^{-1}\}$ for $m \geqq 2$. E_m is compact in X. For if $\{x_n\} \in E_m$, we may suppose, since $\bar{X}$ is compact, that $x_n \to x_0 \in \bar{X}$. If $x_0 \in \partial X$, there is a nonconstant f, holomorphic in a neighborhood of $\bar{X}$ such that $|f(x_0)| = \|f\|_\infty = 1$ [9]. In particular, $|f(p_0)| < 1$. Replacing f by $h \circ f$ where h is the conformal map of the unit disc taking $f(p_0)$ to 0, we may have in fact $f(p_0) = 0$. Thus $|f(x_n)| \leqq 1 - m^{-1}$ for all n. Taking limits, $|f(x_0)| \leqq 1 - m^{-1}$, a contradiction. Thus we must have $x_0 \in E_m$, so E_m is compact.

$X = \cup E_m$. Let $x \in X$. If x were in no E_m, we could find a sequence $\{f_n\} \subset \mathcal{O}(X)$, $\|f_n\|_\infty \leqq 1$, $|f_n(x)| \geqq 1 - m^{-1}$ and $f_n(x_n) = 0$ with $x_n \in O$. We may suppose that $x_n \to x_0 \in X$ since $\bar{O}$ is compact, and by Vitali's theorem, we may suppose that $f_n \to f \in \mathcal{O}(X)$, the convergence being uniform on compact subsets of X. Then $\|f\| \leqq 1$ and $f(x_0) = 0$, but $|f(x)| \geqq \lim |f_n(x)| \geqq 1$, so $|f(x)| = 1$, a contradiction, since a nonconstant function cannot attain its maximum.

E_m is G-invariant. Let $x \in E_m$, $g \in G$. Let $f \in \mathcal{O}(X)$; $\|f\|_\infty \leqq 1$ and $f(p) = 0$ for $p \in O$. Then $\|f \circ g\|_\infty \leqq 1$, $f \circ g(g^{-1}(p)) = 0$ and $g^{-1}(p) \in O$, so, $|(f \circ g)(x)| \leqq 1 - m^{-1}$, since $x \in E_m$. Thus $|f(g(x))| \leqq 1 - m^{-1}$ for every such f, so $g(x) \in E_m$.

Now, let $\{g_n\}$ be a sequence in G. On any compact subset C of X, $\{g_n\}$, as well as $\{g_n^{-1}\}$ has a uniformly convergent subsequence. For, since $\bar{X}$ has a Stein neighborhood, we can view X as a bounded subset of $\mathbf{C}^N$, so the $\{g_n\}$ becomes N-tuples of bounded functions, and Vitali's theorem applies. By diagonalization, we can find a subsequence $\{h_n\}$ of $\{g_n\}$ such that h_n, h_n^{-1} converge uniformly on each E_m. Let $h = \lim h_n$, $k = \lim h_n^{-1}$. Since $h_n \circ h_n^{-1} = I$ on E_m, $h \circ k = I$ on E_m, so $h\colon X \to X$ is invertible. Thus every sequence in G has a convergent subsequence, so (since G is clearly separable), G is compact.

Now, we return to an $H\Sigma$-manifold Σ: (Σ, B, J) is a compact CR-homogeneous strongly pseudoconvex hypersurface. By Theorem 2.2, if $\dim \Sigma \geqq 5$ there is a normal Stein space $\hat{X}$ with only isolated singularities such that Σ bounds a domain X in X_0. From now on we shall assume that this is the case, allowing $\dim \Sigma = 3$ also. Since the automorphisms $g\colon \Sigma \to \Sigma$ can be viewed as N-tuples of holomorphic functions defined near X, they extend, using Theorem VII D4 of [6], to automorphisms of X. Let S be the singular locus of X_0; we shall assume that $S \neq \emptyset$.

3.2. **Lemma.** *Let G be the identity component of $G(X)$. G leaves S pointwise fixed, and is compact.*

Proof. Let $p \in S$, $g \in G$. Then $g(p)$ is also singular. Thus $G: S \to S$. Since S is a finite set, and G connected, G leaves S pointwise fixed.

By the preceding lemma, since G has fixed points, G is compact.

3.3. **Theorem.** *S consists of only one point $\{p_0\}$. If $p \neq p_0$, the orbit G_p of p under G is a compact hypersurface Σ_p bounding a domain D_p with $x_0 \in D_p$.*

Proof. Let $H = \{p \in X; G_p$ is a compact hypersurface Σ_p bounding a domain $D_p \supset S\}$. By hypothesis, G contains a subgroup $(G(\Sigma))$ which extends analytically across $\partial X = \Sigma$ and is transitive there. By the semicontinuity of orbit dimension, for p near ∂X, $\dim G(\Sigma)_p \geqq 2n-1$. Thus $\dim G_p \geqq 2n-1$. Since X is not an orbit of G, we must have $\dim G_p = 2n-1$, so G_p is a compact hypersurface. Since $\lim G_p = \partial X$ as $p \to \partial X$ (if E_m intersects the interior of G_p, it is contained therein) for p close enough to ∂X, G_p bounds a domain containing S. Thus $H \neq \emptyset$. With the same reasoning, we prove that H is open.

Now we show that H is closed in $X - S$. Let $p_n \in H$, $p_n \to q$. Choose a subsequence $\{p_n\}$ so that $Gp_n \neq Gp_m$ (if that is impossible, then $p_n \in Gp_N$ for $n \geqq N$, so $q \in Gp_N$ also). Now, for each n, Gp_n separates X into two components, one of which is Dp_n. Since $Gp_n \cap Gp_m = \emptyset$, either $Dp_n \subset Dp_m$ or $Dp_m \subset Dp_n$. We can find a subsequence $\{q_n\}$ of $\{p_n\}$ consistent with this ordering, i.e., so that

$$n < m \text{ implies } Dq_n \subset Dq_m \text{ for all } n, m$$

or

$$n < m \text{ implies } Dq_n \supset Dq_m \text{ for all } n, m.$$

In either case the proof will be the same, so we shall assume the second implication holds. Further $q_n \to q$. Let $D_n = Dq_n$, $E = \cap D_n$. Since $D_n \supset S$ for all n, $E \supset S$. Further, D_n is G-invariant.

Now, $Gq = \partial E$. For if $g \in G$, certainly $g(q) \in D_n$ for all n, so $g(q) \in \cap D_n = E$. On the other hand, $g(q) = \lim g(q_n)$, and $g(q_n) \in \partial D_n$. Thus $g(q) \in \partial E$. Now, suppose $x \in \partial D_n = Gq_n$ such that $x_n \to x$. Let $x_n = g_n(q_n)$. By compactness of G, choosing a subsequence, we may assume $g_n \to g \in G$. Then

$$x = \lim x_n = \lim g_n(q_n) = g(q)$$

so $x \in Gq$.

Now, if $\partial E \cap S \neq \emptyset$, then $q \in G(S)$, so $q \in S$. Thus if $q \notin S$, $E \cap S = \emptyset$, so E is a neighborhood of S, and ∂E (being an orbit and thus a manifold) is thus a compact hypersurface bounding the interior of E, which contains S. Thus $q \in H$.

Since H is nonempty, open and closed in $X - S$, $H = X - S$. It remains to show that S contains no more than one point. In the above argument, if q were in S, we would have $\partial E = Gq = \{q\}$, since the singular points are fixed under G. But then $E = \{q\}$ also; since $E \supset S$, we obtain $S = \{q\}$.

§4. *Regularity of the monoidal transform*

Continuing the same discussion, we may now assume (shrinking X if necessary) that Σ bounds a Stein analytic space S with only one singular point $\{p_0\}$, and that G is a compact connected group of automorphisms of $\bar{X}$, leaving $\{p_0\}$ fixed, and transitive on Σ. By monoidal transformation we may replace $\{p_0\}$ by a subvariety of codimension 1. Let $(\hat{X}, \tau)$ be the normalized monoidal transform of X with center at $\{p_0\}$, i.e., $\hat{X}$ is a normal Stein space and $\tau: \hat{X} \to X$ is a proper map such that for $E = \tau^{-1}(p_0)$, (i) $\dim E = \dim \hat{X} - 1$, (ii) $\tau: \hat{X} - E = X - \{p_0\}$.

4.1. **Theorem.** (1) *$\hat{X}$, E are nonsingular.*

(2) *G lifts to a group of automorphisms of $\hat{X}$ such that*

(a) *if $p \notin E$, Gp is a hypersurface*

(b) *if $p \in E$, $Gp = E$.*

Proof. To prove the theorem, we need only work in a neighborhood of p_0 on which G operates. Choose such a neighborhood $D = D_p$ which admits local coordinates (in the sense of definition V A 15 of [*6*]) $z^1, \cdots, z^N$ centered at p_0. Let $g \in G$. Then $g: D \to D$ and $g(p_0) = p_0$. Thus $(z^i \circ g)(p_0) = 0$, so $z^i \circ g$ is in the maximal ideal, generated by $z^1, \cdots, z^N$:

$$z^i \circ g = \sum h^i_j z^j, \quad h^i_j \in \mathcal{O}(D).$$

Now, by definition of the monoidal transformation $V = \{(z, w) \in D \times \mathbf{P}^{N-1};\ z \in D,\ z^i w^j = z^j w^i \text{ for all } i, j\}$ has two branches, one of which is $\{p_0\} \times \mathbf{P}^{N-1}$ and the other is the monoidal transform $\hat{D}$. $\tau: \hat{D} \to D$ is the projection on the first coordinate. E is the intersection of the two branches. Now, define $\hat{g}: D \times \mathbf{P}^{N-1} \to D \times \mathbf{P}^{N-1}$ by

$$\hat{g}(z, w) = (g(z), (\sum h^1_j w^j, \cdots, \sum h^N_j w^j)).$$

Clearly, $\tau \circ \hat{g} = g$, so $\hat{g}$ will be a lift of g to $\hat{D}$, once we verify that $\hat{g}: \hat{D} \to \hat{D}$. It suffices to show that $\hat{g}: V \to V$, for it certainly cannot interchange the branches.

Suppose $(z, w) \in V$. Then $z \in D$ and $z^i w^j = z^j w^i$. We must show $\hat{g}(z, w) = (z', w')$ satisfies the same relations. $z' = g(z) \in D$ and

$$
\begin{aligned}
(z')^i(w')^j &= z^i(\hat{g}(z, w))w^j(\hat{g}(z, w)) \\
&= (z^i \circ g)(z) \cdot \sum_r h_r^j w^r \\
&= \sum_s h_s^i z^s \cdot \sum_r h_r^j w^r \\
&= \sum_{r,s} h_s^i h_r^j (z^s w^r) = \sum_{r,s} h_s^i h_r^j (z^r w^s) \\
&= \sum_r h_r^j z^r \cdot \sum_s h_s^i w^s \\
&= z^j(\hat{g}(z, w))w^i(\hat{g}(z, w)) = (z')^j(w')^i.
\end{aligned}
$$

Now, assertion (2)(a) is trivial since it is true on X. As for (2)(b), clearly, for $p \in E$, $G_p \subset E$. Let $p, q \in E$, and choose $p_n \in X - \{p_0\}$, $p_n \to p$. Then $\cap\, Dp_n = \{p_0\}$, so $\cap\, \hat{D}p_n = E$. Thus there exists $q_n \in \partial \hat{D} p_n = Gp_n$, $q_n \to q$. Let $g_n(p_n) = q_n$, and replace by a subsequence so that $g_n \to g \in G$. Then $g(p) = q$. (2)(b) is proven. Finally (1) follows from the normality of $\hat{X}$. The singular locus of $\hat{X}$ is of codimension at least 2 in $\hat{X}$, so is a proper subvariety of E, as in the singular locus of E. Thus there is a $p_0 \in E$ which is regular for $\hat{X}$ and E both. But $\hat{X}$ is homogeneous along E, so every point on E is regular for $\hat{X}$ and E both. (1) is proven.

§5. *The case* $\Sigma = G$

In this section, as a step toward the general case, we consider an $H\Sigma$-manifold on which a group G of CR-automorphisms acts effectively as well as transitively. This means that the map $G \to \Sigma$, $g \to g(p_0)$ (for some prefixed p_0) is a diffeomorphism, so we can identify Σ with $G: \Sigma = (G, B, J)$, where the CR-structure is invariant under left multiplication on G. Since B is left-invariant, it is generated by left-invariant vector fields $X_1, \cdots, X_k$, $Y_1, \cdots, Y_k$ where $J(X_j) = Y_j, J(Y_j) = -X_j$. Since G is a CR-hypersurface, the $+\,i$ eigenspace A, spanned by the $X_j - iY_j$, is a Lie subalgebra of $g_{\mathbf{C}}$ of codimension 1. An example is given by $SU(2)$, as described in section 2. The purpose of this section is to show that this is the only example of such a G bounding a Stein space.

5.1 **Theorem.** *Suppose that G is as described above, and the CR-structure is induced by the embedding $G \to \partial X$, where X is a strongly pseudoconvex complex manifold with nonempty exceptional set E. Then G is $SU(2)$ or $SO(3)$ and X is the monoidal transform of the unit ball in $\mathbf{C}^2$, or that modulo reflection in the origin.*

Proof. Let $N \to E$ be the holomorphic normal bundle of E in X. G acts on the holomorphic tangent bundle $T(X)$ by its differential, and since E is an orbit of G, $T(E)$ is invariant. Thus there is induced a representation of G on $T(X)|E/T(E) = N$, which we denote by dg.

Fix $p_0 \in E$ and let J be the isotropy group of p_0. By hypothesis $\dim_{\mathbf{R}} G = \dim_{\mathbf{R}} E + 1$, so J is one-dimensional. If $\Lambda: J \to \mathbf{C}^*$ is the representation

$$\Lambda(g) = dg(p_0) | N_{p_0},$$

then N is the bundle induced by Λ:

$$\begin{array}{ccccc} J & \longrightarrow & G & \longrightarrow & E \\ \downarrow \Lambda & & \downarrow & & \\ \mathbf{C}^* & \longrightarrow & N & \longrightarrow & E. \end{array}$$

Thus, the adjoint action of J is just the circle action, so is holomorphic on N.

Let $v_0 \in N_{p_0}$, $v_0 \neq 0$, and $\Sigma_0 = Gv_0$. Since G is compact, Σ_0 is compact; since $N \to E$ is not differentially trivial, Σ_0 is of codimension less than 2, so Σ_0 is a hypersurface. Thus the map $g \to dg(v_0)$ is a covering map of G onto Σ_0, so G inherits from Σ_0 a new structure of $H\Sigma$-manifold, under left translation. In particular, if $A_0 \subset T_{\mathbf{C}}(\Sigma_0)$ is the bundle of holomorphic vectors, $a_0 = dg^{-1}(A_{0,v_0})$ is a Lie subalgebra of $g_{\mathbf{C}}$. Let j be the infinitesimal generator of J on G. Since the differential of the projection $N \to E$ is nonsingular on A_0, $d(dg)_e(j) \notin A_{0,v_0} \oplus \bar{A}_{0,v_0}$, so $j \notin a_0 \oplus \bar{a}_0$. Thus $g_{\mathbf{C}} = \mathbf{C}j \oplus a_0 \oplus \bar{a}_0$. Since AdJ preserves the CR-structure on Σ_0, and thus also on G, $[j, a_0] \subset a_0$, $[j, \bar{a}_0] \subset \bar{a}_0$. The algebraic situation is forcing: we now show that J is a maximal torus in G (this is a very special case of Wang's theorem that G/J is algebraic if and only if J centralizes a torus [2]).

Suppose $k \in g_{\mathbf{C}}$, $[k, j] = 0$. We shall show that $k \in \mathbf{C}j$. Let $k = \alpha j + \beta X + \bar{\beta}\bar{X}$; $\alpha \in \mathbf{R}$, $\beta \in \mathbf{C}$, $X \in a_0$.

$$0 = [j, k] = \beta[j, X] + \bar{\beta}[j, \bar{X}],$$

and $[j, X] \in a_0$. Let $X' \in a_0$:

$$0 = [X', [j, k]] = \beta[X', [j, X]] + \bar{\beta}[X', [j, \bar{X}]]. \tag{5.2}$$

Let $\Lambda \in g^*$ be such that $\Lambda^\perp = \operatorname{Re} a_0$, $\Lambda(j) = 1$. Since the CR-structure is strongly pseudoconvex, we know that the Levi form $L(X,Y) = i\Lambda([X,\bar{Y}]) = -i[j,[X,\bar{Y}]]$ is definite hermitian on a_0. Thus, applying Λ to (5.2):

$$0 = \bar{\beta} L(X',[j,X]).$$

This is true for all $X' \in a_0$, so $[j,X] = 0$. But then also $[j,\bar{X}] = 0$, so by the Jacobi identity

$$L(X,\bar{X}) = -i[j,[X,\bar{X}]] = 0$$

and thus, since L is definite, $X = 0$, so $k = \alpha j$. Since $G = J$ is excluded ($\dim_R G \geqq 3$), g has no center so is semisimple, and j generates a maximal torus J in G. Since $\dim J = 1$, we conclude that G is $SU(2)$ or $SO(3)$, [7], and $g_{\mathbf{C}} = \mathbf{C}j \oplus a_0 \oplus \bar{a}_0$ is its root space decomposition.

In case $G = SU(2)$, the realization of G as the unit sphere in $\mathbf{C}^2$ induces this CR-structure, thus $N \to E$ is the Hopf bundle $Q\mathbf{C}^2 \to \mathbf{P}^1$. Thus, blowing E down to $\{p_0\}$ on X, we see that p_0 is a regular point (since its tangent cone is $\mathbf{C}^2$), and G acts as a group of inner transformations at p_0 whose linear part is the group of unitary transformations $SU(2)$. According to Cartan's theorem [*1*], a change of coordinates realizes G as this group, and the theorem is proven.

The case $G = SO(3)$ reduces to the above. Differentiably, $X - E = SO(3) \times (0,1]$. Let $\tilde{X} = SU(2) \times (0,1]$, $\pi\colon \tilde{X} \to X - E$ the obvious 2-sheeted covering. Endow $\tilde{X}$ with the unique complex structure making π a holomorphic map. Now the natural action of $SU(2)$ on $\tilde{X}$ is locally the lift of the $SO(3)$ action on $X - E$, so $SU(2)$ acts holomorphically and effectively on $\tilde{X}$, and the orbits are strongly pseudoconvex hypersurfaces. Let X_0 be X with E blown down to a point $\{p_0\}$ and let $\tilde{X}_0$ be $\tilde{X} \cup \{q_0\}$, topologized so that $\pi\colon \tilde{X}_0 \to X_0$ extended by $\pi(q_0) = p_0$ is continuous. $\tilde{X}_0$ is a β-analytic space in the sense of Grauert and Remmert [*4*], and by their theorem carries the structure of a normal analytic space, and the action of $SU(2)$ extends to $\tilde{X}_0$. Now we are in the first case, and we can conclude that $\tilde{X}_0$ is a ball in $\mathbf{C}^2$, $SU(2)$ acts in its natural way, and $\pi\colon \tilde{X}_0 \to X_0$ is a 2-sheeted covering map with p_0 the branch point. The theorem is proven.

§6. *The general case*

We now return to the situation as it was at the end of section 4. X is a strongly pseudoconvex manifold, with exceptional submanifold E of codimension 1. G is a compact group of biholomorphic mappings of $\bar{X}$, which is transitive on $\partial X = \Sigma$. We have that E is invariant under G and G is transitive on E. Thus G acts, by the differential, on the normal bundle N of the embedding of E in X. We may assume that N is endowed with a G-invariant metric. To show that X is biholomorphically G-equivalent to N, we must find the fibering of X over E; a reduction to the case of section 5 accomplishes that.

6.1. **Theorem.** *There is a biholomorphic G-map of X into N mapping Σ onto the unit circle bundle.*

Proof. For $p \notin E$, its G-orbit, Gp is a hypersurface; for $p \in E$, $Gp = E$. Let $p_0 \in \Sigma$, I = isotropy group of p_0, and K the set of points whose isotropy group contains I:

$$I = \{g \in G;\, g(p_0) = p_0\},\ K = \{x \in X;\, g(x) = x \text{ for } g \in I\}.$$

Since K is the fixed point set of a family of transformations of X, it is an analytic set in X. Let N be the normalizer of I: $N = \{g \in G;\, g^{-1}Ig \subset I\}$.

If $I = \{e\}$, we are in the situation of section 5 and the theorem is proven. For $I \neq \{e\}$, we can conclude that $\dim K \geqq 1$. For, when the orbits are equidimensional, isotropy groups of neighboring points are conjugate. Thus if Σ' is an orbit near Σ, and $p \in \Sigma'$ is near p_0, $I_p = g^{-1}Ig$ for some $g \in G$. Then $I_{g(p)} = I$, and $g(p) \in K$. Since K thus intersects every orbit near Σ, $\dim K \geqq 1$.

Now, K is N-invariant: let $g \in N$, $x \in K$, $h \in I$. Then h fixes $g(x)$: $h(g(x)) = g(h'(x))$ for some $h' \in I$, but then $h'(x) = x$, so $h(g(x)) = g(x)$. Thus $I_{g(x)} \supset I$, so $g(x) \in K$.

For any orbit Σ', N is transitive on $K \cap \Sigma'$; in fact if $x, g(x) \in K$, then $g \in N$: let $h \in I$. $g^{-1}hg(x) = g^{-1}g(x)$, so $g^{-1}hg \in I$ also.

Thus we see that $K - E$ is a manifold, since each orbit is of real codimension 1, so contains a regular point of K. Further, N/I is a group of transformations on K, which is transitive and effective on $K \cap \Sigma$. If $\dim K \geqq 2$, $K \cap E$ is also exceptional in K, so we are in the case of section 5 and we conclude that there is a holomorphic N-map $\tau_0 : K \to K \cap E$ representing K as a holomorphic line bundle over $K \cap E$. Let W_0 be the holomorphic vector field tangent to the fibers which generates the action of $\mathbf{C}^*$. If $\dim_{\mathbf{C}} K = 1$, K is a disc and N/I acts as the circle; the existence of τ_0 and W_0 are self-evident.

For $p \in X$, choose $g \in G$ so that $g(p) \in K$. Let $\tau(p) = g^{-1}\tau_0 g(p)$. This $\tau: X \to E$ is well-defined. For if $g(p) \in K$ also, then $g'g^{-1}: g(p) \to g'(p)$, both in K, so $g'g^{-1} \in N$. Since τ_0 is an N-map, $\tau_0(g'g^{-1}) = (g'g^{-1})\tau_0$, so

$$\tau_0 g'(p) = g' g^{-1} \tau_0(g(p)),$$
$$g'^{-1} \tau_0 g'(p) = g^{-1} \tau_0 g(p).$$

Similarly $W(p) = dg^{-1}(W_0(g(p)))$ is well-defined, since W_0 is also N-invariant. τ, W are holomorphic since they are invariant under a transitive group of holomorphic mappings. Thus, by Proposition 2.3, Σ can be realized as a circle bundle in some complex line bundle over E. But any two realizations of Σ as a hypersurface are biholomorphic (that is, the CR-correspondence of the two Σ's extends to a biholomorphic mapping between the domains they bound). Thus X is this complex line bundle over E (near E), from which we conclude that $X \subset N$.

This theorem as stated in the introduction now easily follows; all that is needed is to point out that since E arises from monoidal transformation, it is given as a subvariety of a projective space, so is projective. This proof includes half the theorem of Morimoto and Nagano. For suppose that Σ is an $H\Sigma$ space in a Stein manifold M under a compact group of CR-automorphisms G. Then Σ, being a hypersurface, must have one strongly pseudoconvex point, being homogeneous, all points are strongly pseudoconvex. It follows that Σ bounds a domain in M. As in section 3, since G is compact, we can conclude that each orbit but one is a hypersurface. If the exceptional orbit is a point, our arguments apply and give the result that Σ is the sphere in $\mathbf{C}^n$, for only the line bundle $\mathbf{C}^n \to \mathbf{P}^{n-1}$ is the monoidal transform of a regular point. However, if the exceptional orbit is not a point, we must refer to their argument.

NOTES

1. L. Nirenberg has recently constructed an example showing this to be false in the C^∞ case.
2. P. Griffiths has shown me a proof of this.

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BRANDEIS UNIVERSITY